Parrondo's Ratchet II

Analytic Theory of Some Random Walks on $\mathbb{Z}$

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Introduction. The history of what I have called "Parrondo's phenomenon"—a subject most commonly known in the by now quite extensive literature as "Parrondo's Paradox" or "Parrondo's Game"—traces to a remark made by the Spanish physicist Juan M. R. Parrondo in the course of a presentation on the "Efficiency of Brownian motors" at a Workshop on Complexity and Chaos that took place in Torino, Italy in July, 1996. In the long introduction to a companion essay[1] I have sketched the train of thought that—over the course of more than twenty centuries—led from the philosophical speculations of Democritus to 19$^{\text{th}}$ Century attempts to address the question "Are atoms real?", to the birth of Maxwell's Demon, and finally, at the beginning of the 20$^{\text{th}}$ Century, to the Einstein-Smoluchowski theory of Brownian motion and to Smoluchowski's invention (1912) of the "Brownian ratchet," which is a mindless variant of Maxwell's Demon intended to demonstrate how one might harness the molecularly-induced Brownian motion of a paddle wheel to do work (and why the simplest version of any such device is in fact bound ultimately to fail). Chapter 46 ("Ratchet and Pawl") in the first volume of Feynman's *Lectures on Physics* (1963) is based upon a lecture in which Feynman made pedagogical use (without attribution) of Smoluchowski's hypothetical device in an effort to make intuitively plausible some of the fundamentals of thermodynamics. In 1996, Parrondo published a criticism of some aspects of Feynman's argument,[2] and it was that work that he discussed at the workshop mentioned above. The train of thought that led from the thermodynamics of "Feynman's ratchet" to the discovery of "Parrondo's Paradox" has, so far as I am aware, never been detailed in print.

While the ratchets encountered in ratchet wrenches and clockworks are—like those contemplated by Smoluchowski and Feynman—circular (and have always a finite number of teeth), those encounted in auto jacks are linear, and could in principle be infinitely long and have an infinite number of teeth. It was with ratchets of the former type in mind that in *Parrondo's ratchet* I was

[1] *Parrondo's Ratchet I* (January 2014).

[2] Juan M. R. Parrondo & Pep Español, "Criticism of Feynman's analysis of the ratchet as an engine," AJP **64**, 1125-1130 (1996).

led to consider random walks on the simplest non-trivial cyclic graph (order 3). It is shown there that when the defining parameters are set to certain values a certain property shared by the walks generated by Markov matrices $\mathbb{A}$ and $\mathbb{B}$ is counterintuitively/"paradoxically" not shared by the walk generated by $\mathbb{AB}$.

Parrondo and collaborators have, on the other hand, found it more natural to draw their motivating imagery from unbounded *linear* ratchets, which leads me here to consider random walks on the linear lattice $\mathbb{Z}$. It must, however, be admitted from the outset that random walks, whether on cyclic graphs or $\mathbb{Z}$, are not really very "ratchet-like," for they lack any feature corresponding to the pawl, which is an essential feature of any real ratchet system. Such models refer more naturally to games of chance (coin-flipping games), and it is in game-theoretic terms that results in this field are most commonly formulated.[3]

Here we study nearest-neighbor walks on $\mathbb{Z}$ (stand-in-place forbidden) that are generated by Markov matrices $\mathbb{A}$ and $\mathbb{B}$ and their relationships to the walk generated by their compose $\mathbb{C} = \mathbb{AB}$. The structure of $\mathbb{B}$ (of which $\mathbb{A}$ is a degenerate special case) will be designed to preserve the period-3 character of walks on the cyclic graph of order 3. The analysis is made relatively more complicated by the circumstance that stochastic state vectors and the Markov matrices that act upon them are now infinite-dimensional. To deal with that problem I adopt a clever line of argument devised by Ray Mayer.[4] We will witness the emergence of "Parrondo's phenomenon" after only a few steps, but—surprisingly, and contrary to the impression conveyed by the literature—will find that it evaporates asymptotically.

Simple theory of A-walks. A walker advances to the right with site-independent probability a, retreats to the left with probability $A = 1 - a$. During the course of an n-step walk the walker—assumed to have departed from the origin—takes k steps to the right and $n - k$ steps to the left, arriving finally (after n decisions selected from a total of 2^n options) at site

$$\# = k - (n - k) = 2k - n \quad : \quad k = 0, 1, 2, \ldots, n$$

which, we note, is even or odd according as n is even or odd.[5] The probability that he arrives at that site is

$$p_{n,k} = \tbinom{n}{k} a^k (1-a)^{n-k} \quad : \quad \sum_{k=0}^{n} p_{n,k} = 1$$

[3] See G. Harmer, D. Abbott, P. Taylor & J. Parrondo, "Brownian ratchets and Parrondo's games," Chaos **11**, 705 (2001); E. Key, M. Klosek & D. Abbott, "On Parrondo's paradox: how to construct unfair games by composing fair games," arXiv:math/020615v1 (15 Jun 2002) and papers cited there.

[4] Personal communication (10-page note taped to my office door on 6 November 2014, responsive to a question posed on 10 October).

[5] Nearest-neighbor walks on $\mathbb{Z}$ "blink" in the sense suggested by the following figure

and discussed on pages 8-9 of *Parrondo's Ratchet I*.

and the expected mean of the set of endpoints generated by many such walks (expected gain if he were winning/losing pennies by flip of a loaded coin) is

$$S_n(a) = \sum_{k=0}^{n}(2k-n)\,p_{n,k} = n(2a-1) \tag{1}$$

—quite as anticipated: the expected gain per flip is $a - A = 2a - 1$ so by a simple scaling principle we expect the gain after n flips to be $n(2a-1)$. The gambler can expect to break even (the walker to end up where he began, though this is strictly possible only if n is even) if $a = \frac{1}{2}$. The expected rate of gain of the gambler ("velocity" of the mean position of an ensemble of walkers, analog of the asymptotic "probability current" for walkers on a cyclic graph) is

$$J(a) \equiv \tfrac{d}{dn}S_n(a) = 2a - 1 = S_1(a)$$

These elementary results will serve as benchmarks that we will use to check the accuracy of results obtained in more complex situations.

Theory of B-walks: Part 1. Analysis in this case is made relatively complicated by the circumstance that the next-step probabilities are now site-dependent. Parrondo's assumption that they proceed $x, y, y, x, y, y, x, y, y, \ldots$ tends to disrupt the symmetry of the argument; I therefore will assume that they proceed $x, y, z, x, y, z, x, y, z, \ldots$ and recover Parrondo's assumption as a special case, after the fact. The 3^{rd}-order periodicity of that progression will become a dominant feature of the discussion.

Let $\mathbb{B}$—sometimes denoted $\mathbb{B}_{x,y,z}$—be the ∞-dimensional matrix of which the central portion is shown below:

$$\mathbb{B} = \begin{pmatrix}
0 & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Z & 0 & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & Y & 0 & z & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & X & 0 & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & Z & 0 & x & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & Y & \color{red}{0} & z & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & X & 0 & y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Z & 0 & x & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Y & 0 & z & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & X & 0 & y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Z & 0
\end{pmatrix}$$

Here $\color{red}{0}$ marks the center of the matrix, the dots $\cdot$ are to be read as zeros, and $X \equiv 1 - x$, $Y \equiv 1 - y$, $Z \equiv 1 - z$. The columns of $\mathbb{B}$ sum to unity and its elements fall within the unit interval $[0, 1]$, so $\mathbb{B}$ is Markovian. Assume that the walker stands initially at the origin; *i.e.*, that his initial state is described by the stochastic vector

$$\boldsymbol{p}_0 = \begin{pmatrix} \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ {\color{red}1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \end{pmatrix}$$

After n steps his stochastic state has become

$$\boldsymbol{p}_n = \mathbb{B}^n \boldsymbol{p}_0$$

His expected position (mean of the positions achieved by many n-step walks) is

$$S_n(x,y,z) = (\boldsymbol{w}, \boldsymbol{p}_n) = (\boldsymbol{w}, \mathbb{B}^n \boldsymbol{p}_0) \quad \text{where} \quad \boldsymbol{w} = \begin{pmatrix} \cdot \\ 4 \\ 3 \\ 2 \\ 1 \\ {\color{red}0} \\ -1 \\ -2 \\ -3 \\ -4 \\ \cdot \end{pmatrix} \tag{2}$$

To ascend thus (by what I call the **naive method**) to order n one must (to avoid boundary errors) work with $\mathbb{B}$-matrices of dimension $\nu \geqslant 2n+1$. It is by this means (working with $\nu = 15$) that I obtained the results reported below:

$$\begin{aligned}
S_4(x,y,z) = {} & -4+4x+2y-2xy+2x^2y+2xy^2-2x^2y^2+2z-2x^2z+4xyz-2xy^2z \\
& +2z^2-4xz^2+2x^2z^2-2yz^2+2xyz^2 \\
S_5(x,y,z) = {} & -5+4x+2x^3+2y+2xy-4x^3y+2y^2-4xy^2+2x^3y^2+4z-4xz+4x^2z \\
& -4x^3z+2xyz-2x^2yz+4x^3yz-4y^2z+8xy^2z-2x^2y^2z+2xz^2-4x^2z^2 \\
& +2x^3z^2-2yz^2+4xyz^2-2x^2yz^2+2y^2z^2-4xy^2z^2 \\
S_6(x,y,z) = {} & -6+4x+4x^2-2x^3+4y-8x^2y+6x^3y-2xy^2+8x^2y^2-6x^3y^2+2xy^3-4x^2y^3 \\
& +2x^3y^3+4z-6x^2z+2x^3z-4yz+8xyz+8x^2yz-4x^3yz+2y^2z+2xy^2z \\
& -10x^2y^2z+2x^3y^2z-4xy^3z+4x^2y^3z+2xz^2-4x^2z^2+2x^3z^2+4yz^2 \\
& -10xyz^2+8x^2yz^2-2x^3yz^2-4y^2z^2+2xy^2z^2+2xy^3z^2+2z^3-6xz^3 \\
& +6x^2z^3-2x^3z^3-4yz^3+8xyz^3-4x^2yz^3+2y^2z^3-2xy^2z^3
\end{aligned}$$

$$\begin{aligned}S_7(x,y,z) = {}&-7+6x+2x^2+4y-4xy+6x^2y-6x^4y+8xy^2-14x^2y^2+6x^4y^2+2y^3-8xy^3\\&+8x^2y^3-2x^4y^3+4z-6x^2z+8x^3z-6x^4z+8xyz+2x^2yz-16x^3yz\\&+12x^4yz+2y^2z-20xy^2z+20x^2y^2z+8x^3y^2z-6x^4y^2z-6y^3z+20xy^3z\\&-14x^2y^3z+4z^2-12xz^2+18x^2z^2-16x^3z^2+6x^4z^2-6yz^2+12xyz^2\\&-16x^2yz^2+16x^3yz^2-6x^4yz^2-4y^2z^2+20xy^2z^2-18x^2y^2z^2+6y^3z^2\\&-16xy^3z^2+6x^2y^3z^2-2z^3+8xz^3-12x^2z^3+8x^3z^3-2x^4z^3+2yz^3\\&-4xyz^3+2x^2yz^3+2y^2z^3-8xy^2z^3+6x^2y^2z^3-2y^3z^3+4xy^3z^3\end{aligned}$$

But raising ever larger matrices to ever higher powers becomes rapidly unfeasible, and it is clearly not possible by such naive means to construct estimates of the form assumed by $S_n(x, y, z)$ as n becomes asymptotically large.

Plot,[6] for ascending values of n, the "null surfaces" that result from setting $S_n(x, y, z) = 0$. It becomes apparent (see PLATES 1 & 3) that those surfaces rapidly converge to a limit... which raises two interrelated questions:

- How does one construct a description of the asymptotic break-even condition described by that limiting null surface?
- How, by inspection of multinomials such as those shown above, does one recognize that (at last within the unit cube) they tend toward "saying the same thing"? This I call the "polynomial similarity problem."

The **alternative method** devised by Ray Mayer[4]—developed below—provides a sharp answer to the first question, and some insight into the answer to the second.

Theory of B-walks: Part 2. The naive method was seen at (2) to proceed from

$$S_n(x, y, z) = (\boldsymbol{w}, \mathbb{B}^n \boldsymbol{p}_0)$$

Mayer's method proceeds from the obviously equivalent equation

$$S_n(x, y, z) = (\boldsymbol{p}_0, \mathbb{D}^n \boldsymbol{w}) \quad \text{where} \quad \mathbb{D} \equiv \mathbb{B}^{\mathsf{T}} \tag{3}$$

Introduce vectors

$$\boldsymbol{F}_1 = \begin{pmatrix} \cdot \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \cdot \end{pmatrix}, \quad \boldsymbol{F}_2 = \begin{pmatrix} \cdot \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \cdot \end{pmatrix}, \quad \boldsymbol{F}_3 = \begin{pmatrix} \cdot \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \cdot \end{pmatrix}, \quad \boldsymbol{F}_0 = \sum_{k=1}^{3} \boldsymbol{F}_k = \begin{pmatrix} \cdot \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdot \end{pmatrix}$$

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[6] Use the *Mathematica* v7 command

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ContourPlot3D[Sn[x,y,z] == 0, {x,0,1}, {y,0,1}, {z,0,1}]
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whose periodicity mimics that of $\mathbb{B}$. Matrix multiplications supply

$$\left.\begin{array}{lll} \mathbb{D}\boldsymbol{F}_1 = & Y\boldsymbol{F}_2 + z\boldsymbol{F}_3 & = g_1(x)\boldsymbol{F}_1 + g_2(y)\boldsymbol{F}_2 + g_3(z)\boldsymbol{F}_3 \\ \mathbb{D}\boldsymbol{F}_2 = & x\boldsymbol{F}_1 \qquad + Z\boldsymbol{F}_3 & = g_3(x)\boldsymbol{F}_1 + g_1(y)\boldsymbol{F}_2 + g_2(z)\boldsymbol{F}_3 \\ \mathbb{D}\boldsymbol{F}_3 = & X\boldsymbol{F}_1 + y\boldsymbol{F}_2 & = g_2(x)\boldsymbol{F}_1 + g_3(y)\boldsymbol{F}_2 + g_1(z)\boldsymbol{F}_3 \end{array}\right\} \tag{4}$$

where we note that the functions

$$\begin{array}{lcl} g_1(u) = 0 & : & \text{abbreviated } g_{1,u} \\ g_2(u) = U \equiv 1-u & : & \text{abbreviated } g_{2,u} \\ g_3(u) = u & : & \text{abbreviated } g_{3,u} \end{array}$$

sum to unity, and that $\mathbb{D}(\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3) = \boldsymbol{F}_0$. Also (by matrix multiplication)

$$\mathbb{D}\,\boldsymbol{w} = \begin{pmatrix} \cdot \\ 5y+3Y \\ 4x+2X \\ 3z+1Z \\ 2y+0Y \\ 1x-1X \\ 0z-2Z \\ -1y-3Y \\ -2x-4X \\ -3z-5Z \\ \cdot \end{pmatrix}$$

which we find with a little inspired tinkering can be written

$$\mathbb{D}\,\boldsymbol{w} = \boldsymbol{w} + \boldsymbol{G}_1 \tag{5}$$

$$\begin{aligned} \boldsymbol{G}_1 &= (2x-1)\boldsymbol{F}_1 + (2y-1)\boldsymbol{F}_2 + (2z-1)\boldsymbol{F}_3 \\ &= f(x)\boldsymbol{F}_1 + f(y)\boldsymbol{F}_2 + f(z)\boldsymbol{F}_3 \\ &\equiv \alpha_1\boldsymbol{F}_1 + \beta_1\boldsymbol{F}_2 + \gamma_1\boldsymbol{F}_3 \end{aligned} \tag{6}$$

with $f(u) = 2u - 1 = u - (1-u) = u - U$.[7] Iteration of (5) gives

[7] We note in passing that—from the definitions of $\boldsymbol{w}$ and the $\boldsymbol{F}$-vectors and the placement of the solitary non-zero element of the stochastic vector $\boldsymbol{p}_0$ (which reflects our easily altered working assumption that the walker departs from the origin)—it follows from (5) that

$$S_1(x,y,z) = (\boldsymbol{p}_0, \mathbb{D}\,\boldsymbol{w}) = \alpha_1 = 2x - 1 = S_1(x)$$

where $S_1(x)$ was defined at (1): the first step of a B-walker is equivalent to the first step of an A-walker with $a = x$.

$$\left.\begin{aligned} \mathbb{D}\ \boldsymbol{w} &= \boldsymbol{w}+\boldsymbol{G}_1 &&= \boldsymbol{w}+\boldsymbol{H}_1\\ \mathbb{D}^2\boldsymbol{w} &= \boldsymbol{w}+\boldsymbol{G}_1+\boldsymbol{G}_2 &&= \boldsymbol{w}+\boldsymbol{H}_2\\ \mathbb{D}^3\boldsymbol{w} &= \boldsymbol{w}+\boldsymbol{G}_1+\boldsymbol{G}_2+\boldsymbol{G}_3 &&= \boldsymbol{w}+\boldsymbol{H}_3\\ &\vdots\\ \mathbb{D}^n\boldsymbol{w} &= \boldsymbol{w}+\boldsymbol{G}_1+\boldsymbol{G}_2+\cdots+\boldsymbol{G}_n &&= \boldsymbol{w}+\boldsymbol{H}_n \end{aligned}\right\}\tag{7}$$

where

$$\boldsymbol{G}_n = \mathbb{D}\,\boldsymbol{G}_{n-1} = \mathbb{D}^{n-1}\boldsymbol{G}_1$$

Bringing (4) to (6), we have

$$\begin{aligned}\boldsymbol{G}_2 = \alpha_2\boldsymbol{F}_1+\beta_2\boldsymbol{F}_2+\gamma_2\boldsymbol{F}_3 = \mathbb{D}\,\boldsymbol{G}_1 = \quad &\alpha_1\cdot\{g_{1,x}\boldsymbol{F}_1+g_{2,y}\boldsymbol{F}_2+g_{3,z}\boldsymbol{F}_3\}\\ &+\beta_1\cdot\{g_{3,x}\boldsymbol{F}_1+g_{1,y}\boldsymbol{F}_2+g_{2,z}\boldsymbol{F}_3\}\\ &+\gamma_1\cdot\{g_{2,x}\boldsymbol{F}_1+g_{3,y}\boldsymbol{F}_2+g_{1,z}\boldsymbol{F}_3\}\end{aligned}$$

giving

$$\begin{pmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{pmatrix} = \mathbb{G}\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix} \quad \text{with} \quad \mathbb{G} = \begin{pmatrix} g_{1,x} & g_{3,x} & g_{2,x}\\ g_{2,y} & g_{1,y} & g_{3,y}\\ g_{3,z} & g_{2,z} & g_{1,z}\end{pmatrix}\tag{8}$$

which is of the form $\boldsymbol{g}_2 = \mathbb{G}\,\boldsymbol{g}_1$ and implies

$$\boldsymbol{g}_n = \mathbb{G}^{n-1}\boldsymbol{g}_1 \quad : \quad n = 1, 2, 3, \ldots\tag{9}$$

Here $\boldsymbol{g}_n$ is a 3-vector (it inherits its dimension from the periodicity of $\mathbb{B}$), assembled from the coordinates (with respect to the $\boldsymbol{F}$-basis) of the ∞-vector $\boldsymbol{G}_n$. Its introduction permits us to pull back from ∞-dimensional theory to a much more tractable 3-dimensional formalism.

Since $\boldsymbol{g}$-space is 3-dimensional, it must be possible to display every $\boldsymbol{g}_n$ as a linear combination of any linearly independent triplet, of which $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ is the most natural candidate. Writing

$$\begin{aligned}\boldsymbol{g}_1 &= 1\boldsymbol{g}_1 + 0\boldsymbol{g}_2 + 0\boldsymbol{g}_3\\ \boldsymbol{g}_2 &= 0\boldsymbol{g}_1 + 1\boldsymbol{g}_2 + 0\boldsymbol{g}_3\\ \boldsymbol{g}_3 &= 0\boldsymbol{g}_1 + 0\boldsymbol{g}_2 + 1\boldsymbol{g}_3\end{aligned}$$

the question arises: What are the coordinates that enter into statements of the form $\boldsymbol{g}_n = a_n\boldsymbol{g}_2 + b_n\boldsymbol{g}_2 + c_n\boldsymbol{g}_3$ $(n = 4, 5, 6, \ldots)$? To answer the question I appeal to some fairly elegant trickery that might be considered extravagant in the present relatively simple context, but that will prove indispensable when we turn to the more complicated *composite* walks to which Parrondo has directed our attention.

More than fifty years ago I devised a way to display the coefficients in the characteristic polynomial of any square matrix $\mathbb{M}$ as algebraic functions of the traces of powers of $\mathbb{M}$.[8] In the 3-dimensional case we have

$$\det(\mathbb{M}-\lambda\,\mathbb{I}) = \sum_{n=0}^{3}\tfrac{1}{n!}Q_n(-\lambda)^{3-n} = \tfrac{1}{6}Q_3 - \tfrac{1}{2}Q_2\lambda + Q_1\lambda^2 - Q_0\lambda^3$$

where

$$\begin{aligned}
Q_0 &= 1\\
Q_1 &= T_1\\
Q_2 &= T_1^2 - T_2\\
Q_3 &= T_1^3 - 3T_1T_2 + 2T_3 = 6\det\mathbb{M}
\end{aligned}$$

and $T_k = \mathrm{tr}\,\mathbb{M}^k$. It follows by the Cayley-Hamilton theorem that

$$\begin{aligned}
\mathbb{M}^3 &= \tfrac{1}{6}Q_3\,\mathbb{I} - \tfrac{1}{2}Q_2\,\mathbb{M} + Q_1\,\mathbb{M}^2\\
&= \tfrac{1}{6}\big(T_1^3 - 3T_1T_2 + 2T_3\big)\,\mathbb{I} - \tfrac{1}{2}\big(T_1^2 - T_2\big)\,\mathbb{M} + T_1\,\mathbb{M}^2\\
&\equiv q_1\,\mathbb{I} + q_2\,\mathbb{M} + q_3\,\mathbb{M}^2
\end{aligned}\tag{10}$$

In the problem at hand the generic $\mathbb{M}$-matrix has become

$$\mathbb{G} = \begin{pmatrix} 0 & x & 1-x\\ 1-y & 0 & y\\ z & 1-z & 0\end{pmatrix}\tag{11}$$

amd *Mathematica* supplies

$$\left.\begin{aligned}
q_1 &= 1-(x+y+z)+(xy+yz+zx) = \det\mathbb{G}\equiv\sigma\\
q_2 &= (x+y+z)-(xy+yz+zx) = 1-\det\mathbb{G} = 1-\sigma\\
q_3 &= 0
\end{aligned}\right\}\tag{12}$$

giving

$$\begin{aligned}
&\boldsymbol{g}_4 = \mathbb{G}^3\boldsymbol{g}_1 = \big[q_1\mathbb{G}^0 + q_2\mathbb{G}^1 + q_3\mathbb{G}^2\big]\,\boldsymbol{g}_1 = q_1\,\boldsymbol{g}_1 + q_2\,\boldsymbol{g}_2 + q_3\,\boldsymbol{g}_3\\
\therefore\;\;&\boldsymbol{g}_5 = q_3q_1\,\boldsymbol{g}_1 + (q_1 + q_3q_2)\,\boldsymbol{g}_2 + (q_3 + q_3q_3)\,\boldsymbol{g}_3\\
&\;\;\vdots
\end{aligned}$$

where the $\boldsymbol{g}$-vectors refer to the $\boldsymbol{F}$-basis. The chain

$$\boldsymbol{g}_1 \xrightarrow[\mathbb{G}]{} \boldsymbol{g}_2 \xrightarrow[\mathbb{G}]{} \boldsymbol{g}_3 \xrightarrow[\mathbb{G}]{} \boldsymbol{g}_4 \xrightarrow[\mathbb{G}]{} \boldsymbol{g}_5 \xrightarrow[\mathbb{G}]{} \cdots$$

is not particularly easy to develop. But when referred to the $\mathbb{G}$-basis it becomes

[8] For a recent account of the old material to which I allude, see "Algorithm for the efficient evaluation of the trace of the inverse of a matrix" (1996), which was written to resolve a problem posed by Richard Crandall.

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow[\mathbb{Q}]{} \begin{pmatrix}0\\1\\0\end{pmatrix} \xrightarrow[\mathbb{Q}]{} \begin{pmatrix}0\\0\\1\end{pmatrix} \xrightarrow[\mathbb{Q}]{} \begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} \xrightarrow[\mathbb{Q}]{} \begin{pmatrix}q_3q_1\\q_1+q_3q_2\\q_2+q_3q_3\end{pmatrix} \xrightarrow[\mathbb{Q}]{} \cdots$$

which is generated by

$$\mathbb{Q} = \begin{pmatrix}0 & 0 & q_1\\1 & 0 & q_2\\0 & 1 & q_3\end{pmatrix} \tag{13}$$

and is much easier to develop, not because $\mathbb{Q}$ is so much simpler than $\mathbb{G}$ (though for Parrondo's composite walks that relative simplicity will become conspicuous) but because $\mathbb{Q}$ is Markovian.[9]

Let us agree to write

$$\boldsymbol{H}_n \equiv \boldsymbol{G}_1 + \boldsymbol{G}_2 + \cdots + \boldsymbol{G}_n = a_n\boldsymbol{G}_1 + b_n\boldsymbol{G}_2 + c_n\boldsymbol{G}_3 \tag{14}$$

$$\boldsymbol{h}_n \equiv \begin{pmatrix}a_n\\b_n\\c_n\end{pmatrix}$$

which is to say: let $\{a_n, b_n, c_n\}$ be the coordinates of $\boldsymbol{H}_n$ relative to the $\boldsymbol{G}$-basis. We have

$$\boldsymbol{h}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \boldsymbol{h}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ \boldsymbol{h}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ \boldsymbol{h}_4 = \begin{pmatrix}1+q_1\\1+q_2\\1+q_3\end{pmatrix},\ \ldots$$

which are seen to be produced by the inhomogeneous iteration rule

$$\begin{aligned}
\boldsymbol{h}_2 &= \mathbb{Q}\boldsymbol{h}_1 + \boldsymbol{h}_1 = (\mathbb{Q}+\mathbb{I})\boldsymbol{h}_1\\
\boldsymbol{h}_3 &= \mathbb{Q}\boldsymbol{h}_2 + \boldsymbol{h}_1 = (\mathbb{Q}\mathbb{Q}+\mathbb{Q}+\mathbb{I})\boldsymbol{h}_1\\
\boldsymbol{h}_4 &= \mathbb{Q}\boldsymbol{h}_3 + \boldsymbol{h}_1 = (\mathbb{Q}\mathbb{Q}\mathbb{Q}+\mathbb{Q}\mathbb{Q}+\mathbb{Q}+\mathbb{I})\boldsymbol{h}_1\\
&\ \vdots\\
\boldsymbol{h}_n &= \mathbb{Q}\boldsymbol{h}_{n-1} + \boldsymbol{h}_1 = \sum_{k=0}^{n-1}\mathbb{Q}^k\boldsymbol{h}_1
\end{aligned} \tag{15}$$

Drawing now upon (3), (7) and (14), we have

$$S_n(x,y,z) = (\boldsymbol{p}_0, \boldsymbol{w}) + (\boldsymbol{p}_0, a_n\boldsymbol{G}_1 + b_n\boldsymbol{G}_2 + c_n\boldsymbol{G}_3)$$

For reasons already explained,[7] the leading term on the right vanishes and the

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[9] That the q-parameters sum to unity has already been remarked. That they range on the unit interval as the $\{x, y, z\}$-parameters range on that interval is more easily demonstrated graphically than analytically. $\mathbb{G}$ is *not* Markovian, though its transpose is.

remaining terms sense only the central elements of the $\boldsymbol{G}$-vectors, so we have

$$S_n(x, y, z) = a_n G_1 + b_n G_2 + c_n G_3 \tag{16}$$

where the central elements in question are

$$\left.\begin{aligned} G_1 &= (2x-1) \\ G_2 &= (2z-1) + 2x(y-z) \\ G_3 &= (2y-1) - 2(yz - zx + xy) + 2x^2(1-y-z) + 4xyz \end{aligned}\right\} \tag{17}$$

which have been read off from the computed values of $\boldsymbol{G}_1 = \mathbb{D}\,\boldsymbol{w} - \boldsymbol{w}$, $\boldsymbol{G}_2 = \mathbb{D}\,\boldsymbol{G}_1$ and $\boldsymbol{G}_3 = \mathbb{D}\,\boldsymbol{G}_2$.

The formula (16) can be used as it stands to obtain explicit results of fairly high order—results that are found to agree precisely with the naively-constructed results reported on pages 4 & 5. To that end one might appeal to *Mathematica* for assistance in evaluating the $\{a_n, b_n, c_n\}$ coordinates that follow from (15). But there is a better way to approach the latter problem—one which will enable us to discuss the *asymptotic* properties of B-walks.

Theory of B-walks: Part 3. We undertake now to construct the generalized spectral resolution[10] of $\mathbb{Q}$, which will permit us to reformulate—in a very useful way—the expression on the right side of (15).

The roots of the cubic polynomial

$$\det(\mathbb{Q} - \lambda\mathbb{I}) = q_1 + q_2\lambda + q_3\lambda^2 - \lambda^3$$

are famously complicated, but if we set $q_2 = 1 - q_1 - q_3$ to render explicit the fact that

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 & q_1 \\ 1 & 0 & 1 - q_1 - q_3 \\ 0 & 1 & q_3 \end{pmatrix} \tag{18}$$

is Markovian we obtain

$$\det(\mathbb{Q} - \lambda\mathbb{I}) = (\lambda - 1)\cdot\big[\lambda^2 + (1 - q_3)\lambda + q_1\big]$$

so have only to solve a quadratic to obtain the complete spectrum.[11] We will have need ultimately of the theory that flows from (18), but for expository convenience I look initally to the theory that results when one brings into play the simplifications (12) that associate with $\mathbb{B}$-walks, writing

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 & \sigma \\ 1 & 0 & 1 - \sigma \\ 0 & 1 & 0 \end{pmatrix} \tag{19}$$

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[10] See pages 6 & 7 of *Parrondo's Ratchet I* and material cited there.

[11] We lose this happy fact if the walk has period greater than three.

From

$$\det(\mathbb{Q}-\lambda\mathbb{I})=(\lambda-1)(\lambda^2+\lambda+\sigma)$$

we obtain eigenvalues

$$\left.\begin{aligned}&\lambda_1=1\\&\lambda_2=-\tfrac{1}{2}(1+\xi)\quad\text{with}\quad\xi=\sqrt{1-4\sigma}\\&\lambda_3=-\tfrac{1}{2}(1-\xi)\end{aligned}\right\}\tag{20}$$

where λ_2 and λ_3 are real if $0\leqslant\sigma<\frac{1}{4}$ and complex conjugates if $\frac{1}{4}<\sigma\leqslant 1$. With the assistance of *Mathematica* we construct column vectors $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$ that are right eigenvectors of $\mathbb{Q}$

$$\mathbb{Q}\,\boldsymbol{u}_k=\lambda_k\boldsymbol{u}_k$$

and row vectors $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$ that are left eigenvectors of $\mathbb{Q}$ (transposed right eigenvectors of $\mathbb{Q}^\mathsf{T}$)

$$\boldsymbol{v}_k\mathbb{Q}=\lambda_k\boldsymbol{v}_k$$

We use those to construct matrices

$$\mathbb{P}_k=\frac{\boldsymbol{u}_k\,\boldsymbol{v}_k}{\boldsymbol{v}_k\,\boldsymbol{u}_k}=\frac{3\times 3\text{ matrix}}{\text{number}}\quad:\quad k=1,2,3$$

which are demonstrably projective

$$\mathbb{P}_k^2=\mathbb{P}_k\quad:\quad k=1,2,3$$

orthogonal

$$\mathbb{P}_j\mathbb{P}_k=\mathbb{O}\quad:\quad j\neq k$$

and complete

$$\mathbb{P}_1+\mathbb{P}_2+\mathbb{P}_3=\mathbb{I}$$

and permit one to write

$$\mathbb{Q}=\lambda_1\mathbb{P}_1+\lambda_2\mathbb{P}_2+\lambda_3\mathbb{P}_3$$

whence

$$\mathbb{Q}^n=\lambda_1^n\,\mathbb{P}_1+\lambda_2^n\,\mathbb{P}_2+\lambda_3^n\,\mathbb{P}_3\tag{21}$$

Mathematica supplies explicit descriptions of the $\mathbb{P}$-matrices that can be written

$$\begin{aligned}\mathbb{P}_1&=D_1{}^{-1}\begin{pmatrix}\sigma&\sigma&\sigma\\1&1&1\\1&1&1\end{pmatrix}\\\mathbb{P}_2&=D_2{}^{-1}\begin{pmatrix}\xi-1+2\sigma&-(\xi-1)\sigma&-2\sigma^2\\-2\sigma&(\xi+1)\sigma&-\frac{1}{2}(\xi+1)^2\sigma\\-(\xi-1)&-2\sigma&(\xi+1)\sigma\end{pmatrix}\\\mathbb{P}_3&=D_3{}^{-1}\begin{pmatrix}\xi+1-2\sigma&-(\xi+1)\sigma&2\sigma^2\\2\sigma&(\xi-1)\sigma&\frac{1}{2}(\xi-1)^2\sigma\\-(\xi+1)&2\sigma&(\xi-1)\sigma\end{pmatrix}\end{aligned}$$

where

$$\begin{aligned} D_1 &\equiv 2+\sigma \\ D_2 &\equiv \xi(1+2\sigma)+(4\sigma-1) \\ D_3 &\equiv \xi(1+2\sigma)-(4\sigma-1) \end{aligned}$$

Returning with (21) to (15) we have

$$\boldsymbol{h}_n = \Big\{ \sum_{k=0}^{n-1} \lambda_1^k \mathbb{P}_1 + \sum_{k=0}^{n-1} \lambda_2^k \mathbb{P}_2 + \sum_{k=0}^{n-1} \lambda_3^k \mathbb{P}_3 \Big\} \boldsymbol{h}_1 \tag{22}$$

Typical low-order results

$$\boldsymbol{h}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{h}_4 = \begin{pmatrix} 1+\sigma \\ 2-\sigma \\ 1 \end{pmatrix}$$

$$\boldsymbol{h}_5 = \begin{pmatrix} 1+\sigma \\ 2 \\ 2-\sigma \end{pmatrix} \qquad \boldsymbol{h}_6 = \begin{pmatrix} 1+2\sigma-\sigma^2 \\ 3-2\sigma+\sigma^2 \\ 2 \end{pmatrix}$$

$$\boldsymbol{h}_7 = \begin{pmatrix} 1+2\sigma \\ 3-\sigma^2 \\ 3-2\sigma+\sigma^2 \end{pmatrix} \qquad \boldsymbol{h}_8 = \begin{pmatrix} 1+3\sigma-2\sigma^2+\sigma^3 \\ 4-3\sigma+3\sigma^2-\sigma^3 \\ 3-\sigma^2 \end{pmatrix}$$

suggest that quite generally

$$\sum \text{elements of } \boldsymbol{h}_n = n$$

Remarkably, all reference to the D-denominators that enter into the construction of the $\mathbb{P}$-matrices, as also to the ξ that appears in the definitions both of those and of the eigenvalues, has evaporated.

Reading now from (16), we have

$$\begin{aligned} S_4(x,y,z) &= a_4 G_1 + b_4 G_2 + c_4 G_3 \\ &= (1+\sigma)G_1 + (2-\sigma)G_2 + G_3 \end{aligned}$$

$$S_5(x,y,z) = (1+\sigma)G_1 + 2G_2 + (2-\sigma)G_3$$

$$S_6(x,y,z) = (1+2\sigma-\sigma^2)G_1 + (3-2\sigma+\sigma^2)G_2 + 2G_3$$

$$S_7(x,y,z) = (1+2\sigma)G_1 + (3-\sigma^2)G_2 + (3-2\sigma+\sigma^2)G_3$$

Those results—reading σ from (12) and $\{G_1, G_2, G_3\}$ from (17)—are found (with *Mathematica*'s assistance) to duplicate precisely the naive results reported on pages 4 & 5. The preceding formulæ provide some insight into how those results acquired their forbidding complexity.

Graphic display (PLATE 1) indicates that the null surfaces defined $S_n(x,y,z)=0$ fall into two qualitatively distinct classes, according as n (small) is odd or even, and that the distinction between those classes diminishes as n becomes larger. Also vividly evident in such figures is the fact that the condition $x=\frac{1}{2}$ serves to inscribe a straight line on those surfaces, as follows from (for example)

$$\begin{aligned}
S_4(\tfrac{1}{2},y,z) &= \tfrac{1}{2}(y+z-1)\big(4+y+z-2yz\big)\\
S_5(\tfrac{1}{2},y,z) &= \tfrac{1}{4}(y+z-1)\big(11+y+z-2yz\big)\\
S_6(\tfrac{1}{2},y,z) &= \tfrac{1}{4}(y+z-1)\big(13+2y+y^2+2z-2yz-4y^2z+z^2-4yz^2+4y^2z^2\big)\\
S_7(\tfrac{1}{2},y,z) &= \tfrac{1}{8}(y+z-1)\big(31+6y-y^2+6z-14yz+4y^2z-z^2+4yz^2-4y^2z^2\big)
\end{aligned}$$

Why the preferential role assigned here to x? Because $S_n(x,y,z)$ refers to the expected mean n-step progress of walkers who depart from the origin or any site indexed $0\bmod 3$, at each of which the next-step probability is set by x. Results appropriate to walkers who depart from any site indexed $1\bmod 3$ are obtained by cyclic permutation $\{x,y,z\}\to\{y,z,x\}$, while $\{x,y,z\}\to\{z,x,y\}$ gives results appropriate to walkers who depart from any site indexed $2\bmod 3$.

Asymptotic implications of the preceding argument. From $\lambda_1=1$ it follows that (22) can for all n be written

$$\boldsymbol{h}_n=\Big\{n\mathbb{P}_1+\sum_{k=0}^{n-1}\lambda_2^k\,\mathbb{P}_2+\sum_{k=0}^{n-1}\lambda_3^k\,\mathbb{P}_3\Big\}\boldsymbol{h}_1$$

And since $|\lambda_{2,3}|<1$ we can for large n write

$$\begin{aligned}
\boldsymbol{h}_n &\sim \Big\{n\mathbb{P}_1+\frac{1}{1-\lambda_2}\,\mathbb{P}_2+\frac{1}{1-\lambda_3}\,\mathbb{P}_3\Big\}\boldsymbol{h}_1\\
&=\Big\{n\mathbb{P}_1+\frac{2}{3+\xi}\;\mathbb{P}_2+\frac{2}{3-\xi}\;\mathbb{P}_3\Big\}\boldsymbol{h}_1\\
&=n\cdot\frac{1}{2+\sigma}\begin{pmatrix}\sigma\\1\\1\end{pmatrix}+\frac{1}{(2+\sigma)^2}\begin{pmatrix}4-\sigma\\ \sigma-1\\ -3\end{pmatrix}\\
&\approx n\cdot\frac{1}{2+\sigma}\begin{pmatrix}\sigma\\1\\1\end{pmatrix}
\end{aligned}\tag{23}$$

which gives

$$\begin{aligned}
S_n(x,y,z)&\approx n\cdot\mathcal{S}(x,y,z)\\
\mathcal{S}(x,y,z)&=\frac{(\sigma G_1+G_2+G_3)}{2+\sigma}\\
&=\frac{6xyz-3\sigma}{2+\sigma}\\
&=\frac{6xyz-3+3(x+y+z)-3(xy+yz+zx)}{3-(x+y+z)+(xy+yz+zx)}\\
&\equiv\frac{\mathcal{P}(x,y,z)}{\mathcal{Q}(x,y,z)}
\end{aligned}\tag{24}$$

This result is—in view of the heavy calculation that went into its derivation—remarkably simple. Note the symmetry of $\mathcal{S}(x, y, z)$; *i.e.*, its invariance under all permutations of its arguments. It is evident graphically (PLATE 2)—and follows analytically from

$$\begin{aligned}\mathcal{S}(\tfrac{1}{2}, y, z) &= (y + z - 1)\cdot f(y, z)\\ \mathcal{S}(x, \tfrac{1}{2}, z) &= (z + x - 1)\cdot f(z, x)\\ \mathcal{S}(x, y, \tfrac{1}{2}) &= (x + y - 1)\cdot f(x, y)\end{aligned}$$

$$f(x, y) = -\frac{3}{5 - x - y + 2xy}$$

—that on the null surface $\mathcal{S}(x, y, z) = 0$ are inscribed (not one but) three straight lines, which partition the surface into two sets of three congruent sectors. Those lines intersect at the center of the unit cube: $\mathcal{S}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$. We infer that the asymptotic mean is the same whatever the walker's point of departure. At $x = y = z = a$ we from (24) obtain

$$S_n(a, a, a) \approx n\cdot(2a - 1) = S_n(a)$$

which we know (compare (1)) to be in fact exact.

Theory of composite C-walks. We turn now to the composite 4-parameter walks generated by $\mathbb{C}_{a,x,y,z} = \mathbb{A}_a\mathbb{B}_{x,y,z}$, though we will in the end have interest only in the 3-parameter walks generated by $\mathbb{C}_{a,x,y,y}$. A central portion of the $\mathbb{C}$-matrix looks like this:

$$\mathbb{C} = \begin{pmatrix} Ax{+}aX & 0 & ay & \cdot & \cdot & \cdot & \cdot \\ 0 & Az{+}aZ & 0 & ax & \cdot & \cdot & \cdot \\ AX & 0 & Ay{+}aY & 0 & az & \cdot & \cdot \\ \cdot & AZ & 0 & Ax{+}aX & 0 & ay & \cdot \\ \cdot & \cdot & AY & 0 & Az{+}aZ & 0 & ax \\ \cdot & \cdot & \cdot & AX & 0 & Ay{+}aY & 0 \\ \cdot & \cdot & \cdot & \cdot & AZ & 0 & Ax{+}aX \end{pmatrix}$$

That $\mathbb{C}$ is Markovian follows from $au + (1-a)u + a(1-u) + (1-a)(1-u) = 1$, and its 3-periodicity is obvious. $\mathbb{C}$ refers to 2-step events, and in two steps a walker can return to his starting point (in two different ways), which is why non-zero terms now appear on the diagonal. But such a walker cannot arrive at a nearest-neighbor point, which accounts for the empty off-diagonals.

The "naive method" remains a computational option, but to ascend to order n one must now use matrices of dimension not less than $\nu = 4n + 1$. It is by such computation (with $\nu = 29$) that—following the pattern and preserving the notation of the argument that began on page 6 (but assigning new meanings now to the symbols)—we are led to write

$$\begin{aligned}\mathbb{D}\boldsymbol{F}_1 &= g_1(a, x)\boldsymbol{F}_1 + g_2(a, y)\boldsymbol{F}_2 + g_3(a, z)\boldsymbol{F}_3\\ \mathbb{D}\boldsymbol{F}_2 &= g_3(a, x)\boldsymbol{F}_1 + g_1(a, y)\boldsymbol{F}_2 + g_2(a, z)\boldsymbol{F}_3\\ \mathbb{D}\boldsymbol{F}_3 &= g_2(a, x)\boldsymbol{F}_1 + g_3(a, y)\boldsymbol{F}_2 + g_1(a, z)\boldsymbol{F}_3\end{aligned}$$

where now

$$\begin{aligned} g_1(a,u) &= Au + aU &&= a + u - 2au \\ g_2(a,u) &= au &&= au \\ g_3(a,u) &= AU &&= 1 - a - u + au \end{aligned}$$

again sum to unity. Moreover

$$\begin{aligned} \mathbb{D}\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 \\ \boldsymbol{G}_1 &= f(a,x)\boldsymbol{F}_1 + f(a,y)\boldsymbol{F}_2 + f(a,z)\boldsymbol{F}_3 \\ &= \alpha_1\boldsymbol{F}_1 + \beta_1\boldsymbol{F}_2 + \gamma_1\boldsymbol{F}_3 \end{aligned}$$

where $f(a,u) = 2a + 2u - 2 = (a - A) + (u - U)$.[12] Again we have equations (7) and (8), the difference being that $\mathbb{G}$ is given now by (compare (11))

$$\mathbb{G} = \begin{pmatrix} a + x - 2ax & 1 - a - x + ax & ax \\ ay & a + y - 2ay & 1 - a - y + ay \\ 1 - a - z + az & az & a + z - 2az \end{pmatrix}$$

which by (10) entails (compare (12))

$$\begin{aligned} q_1 &= (1 - 3a + 3a^2)\big[1 - (x + y + z) + (xy + yz + zx)\big] = \det\mathbb{G} \\ q_2 &= -3a^2 + (3a^2 - a)(x + y + z) - (1 - 3a + 3a^2)(xy + yz + zx) \\ q_3 &= 3a + (1 - 2a)(x + y + z) \end{aligned}$$

Again we have

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 & q_1 \\ 1 & 0 & q_2 \\ 0 & 1 & q_3 \end{pmatrix} \tag{13}$$

which we intend to use as before in (15) to obtain

$$S_n(a,x,y,z) = a_n G_1 + b_n G_2 + c_n G_3$$

And to evaluate the powers of $\mathbb{Q}$ that appear in (15) we will again make use of the spectral decomposition

$$\mathbb{Q} = \lambda_1\mathbb{P}_1 + \lambda_2\mathbb{P}_2 + \lambda_3\mathbb{P}_3$$

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[12] We note in passing that

$$S_1(a,x,y,z) = (\boldsymbol{p}_0, \mathbb{D}\,\boldsymbol{w}) = \alpha_1 = 2(a + x - 1)$$

which in the case $x = a$ supplies the very sensible 2-step result

$$S_1(a,a,y,z) = 2(2a - 1) = 2S_1(a) = S_2(a)$$

Execution of this program is made relatively difficult by the circumstance that the expressions that describe the eigenvalues $\{\lambda_2, \lambda_3\}$, the $\mathbb{P}$-matrices and the central $\boldsymbol{G}$-elements $\{G_1, G_2, G_3\}$ are now significantly more complicated.

We begin by writing

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 & \sigma \\ 1 & 0 & 1-\sigma-\tau \\ 0 & 1 & \tau \end{pmatrix}$$

to render explicit the fact that $\mathbb{Q}$ is Markovian. We have favored $q_1 = \det\mathbb{Q}$ because it is algebraically fundamental, and q_3 because it is the simplest of the q-functions. At $\tau = 0$ we recover (19). The eigenvalues have become

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -\tfrac{1}{2}(1-\tau+\zeta) \quad \text{with} \quad \zeta = \sqrt{1-4\sigma-2\tau+\tau^2} \\ \lambda_3 &= -\tfrac{1}{2}(1-\tau-\zeta) \end{aligned}$$

Proceeding as before we obtain

$$\mathbb{P}_1 = D_1^{-1}\begin{pmatrix} \sigma & \sigma & \sigma \\ 1-\tau & 1-\tau & 1-\tau \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbb{P}_2 = D_2^{-1}\begin{pmatrix} -\frac{1}{2}(\zeta-1+\tau)^2 & -(\zeta-1+\tau)\sigma & -2\sigma^2 \\ -2\sigma+\tau(\zeta-1+\tau) & (\zeta+1+\tau)\sigma & -(\zeta+1-2\sigma-\tau)\sigma \\ -(\zeta-1+\tau) & -2\sigma & (\zeta+1-\tau)\sigma \end{pmatrix}$$

$$\mathbb{P}_3 = D_3^{-1}\begin{pmatrix} +\frac{1}{2}(\zeta+1-\tau)^2 & -(\zeta+1-\tau)\sigma & 2\sigma^2 \\ +2\sigma+\tau(\zeta+1-\tau) & (\zeta-1-\tau)\sigma & -(\zeta-1+2\sigma+\tau)\sigma \\ -(\zeta+1-\tau) & 2\sigma & (\zeta-1+\tau)\sigma \end{pmatrix}$$

where

$$\begin{aligned} D_1 &= 2+\sigma-\tau \\ D_2 &= \zeta\,(1+2\sigma-\tau) + (4\sigma-1+2\tau-\tau^2) \\ D_3 &= \zeta\,(1+2\sigma-\tau) - (4\sigma-1+2\tau-\tau^2) \end{aligned}$$

Mathematica confirms that the $\mathbb{P}$-matrices described above do indeed possess all the properties one expects of a complete set of orthogonal projectors, and that they do indeed collaborate with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ to provide the spectral resolution of $\mathbb{Q}$. Proceeding as before, we bring this information to

$$\boldsymbol{h}_n = \sum_{k=0}^{n-1}\mathbb{Q}^k\boldsymbol{h}_1 = \Big\{n\,\mathbb{P}_1 + \sum_{k=0}^{n-1}\lambda_2^k\,\mathbb{P}_2 + \sum_{k=0}^{n-1}\lambda_3^k\,\mathbb{P}_3\Big\}\boldsymbol{h}_1$$

and (after the replacement $\zeta \rightarrow \sqrt{1-4\sigma-2\tau+\tau^2}$) obtain

$$\boldsymbol{h}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{h}_4 = \begin{pmatrix} 1+\sigma \\ 2-\sigma-\tau \\ 1+\tau \end{pmatrix}$$

$$\boldsymbol{h}_5 = \begin{pmatrix} 1+\sigma+\sigma\tau \\ 2-\sigma\tau-\tau^2 \\ 2-\sigma+\tau^2 \end{pmatrix} \qquad \boldsymbol{h}_6 = \begin{pmatrix} 1+2\sigma-\sigma^2+\sigma\tau^2 \\ 3-2\sigma+\sigma^2+(\sigma-1)(2\tau-\tau^2) \\ 2-2(\sigma-1)\tau-\tau^2 \end{pmatrix}$$

These results—from which the D-denominators have once again magically evaporated (as have all references to ζ), and which again suggest that quite generally $\sum$ elements of $\boldsymbol{h}_n = n$—give back the results reported on page 12 in the limit $\tau \to 0$. They supply (for example)

$$S_4(a,x,y,z) = (1+\sigma)G_1 + (2-\sigma-\tau)G_2 + (1+\tau)G_3$$
$$S_5(a,x,y,z) = (1+\sigma+\sigma\tau)G_1 + (2-\sigma\tau-\tau^2)G_2 + (2-\sigma+\tau^2)G_3$$

where

$$\begin{aligned}\sigma \equiv q_1 &= 1-3a+3a^2-x+3ax-3a^2x-y+3ay-3a^2y+xy-3axy+3a^2xy-z+3az-3a^2z\\ &\quad +xz-3axz+3a^2xz+yz-3ayz+3a^2yz\\ \tau \equiv q_3 &= 3a+x+y+z-2ax-2ay-2az\end{aligned}$$

$$\begin{aligned}G_1 &\equiv \text{central element of } \boldsymbol{G}_1 = \mathbb{D}\,\boldsymbol{w} - \boldsymbol{w}\\ &= -2+2a+2x\\ G_2 &= \text{central element of } \boldsymbol{G}_2 = \mathbb{D}\,\boldsymbol{G}_1\\ &= -2+2a+2ax+2x^2-4ax^2+2y-2ay-2xy+2axy+2axz\\ G_3 &= \text{central element of } \boldsymbol{G}_3 = \mathbb{D}\,\boldsymbol{G}_2\\ &= -2+2a+2a^2x+6ax^2-10a^2x^2+2x^3-8ax^3+8a^2x^3+4ay-4a^2y+2xy-8axy+6a^2xy-2x^2y\\ &\quad +4ax^2y-2a^2x^2y+2y^2-6ay^2+4a^2y^2-2xy^2+6axy^2-4a^2xy^2+2z-4az+2a^2z-2xz\\ &\quad +4axz+2a^2xz-2a^2x^2z-2yz+4ayz-2a^2yz+2xyz-4axyz+4a^2xyz+2axz^2-4a^2xz^2\end{aligned}$$

The expressions that result when that information is assembled are so complicated that they are best allowed to remain within *Mathematica*'s memory; we find, for example, that

$$\begin{aligned}S_3(a,x,y,z) &= -6+6a+2x+2ax+2a^2x+2x^2+2ax^2-10a^2x^2+2x^3-8ax^3+8a^2x^3\\ &\quad +2y+2ay-4a^2y-6axy+6a^2xy-2x^2y+4ax^2y-2a^2x^2y+2y^2-6ay^2\\ &\quad +4a^2y^2-2xy^2+6axy^2-4a^2xy^2+2z-4az+2a^2z-2xz+6axz+2a^2xz\\ &\quad -2a^2x^2z-2yz+4ayz-2a^2yz+2xyz-4axyz+4a^2xyz+2axz^2-4a^2xz^2\\ S_4(a,x,y,z) &= -8+\text{sum of 97 terms, of which the last is } 8a^3xz^3\end{aligned}$$

The results of such calculations, carried to order 7, were found to be in precise agreement with the results of naive calculations done with 29-dimensional $\mathbb{C}$-matrices. It was found, moreover, that (for $n = 1, 2, \dots, 7$)

$$S_n(0,0,0,0) = -2n$$
$$S_n(1,1,1,1) = +2n$$

—quite as one expects: if every step in a series of n double steps is to the left with certainty, then the walker with certainly terminate at $\# = -2n$, and if to the right with certainty then he will with certainly terminate at $\# = +2n$.

Similarly, if every step is left/right with equal probability then he is most likely to end up where he began, and indeed: the preceding results yield

$$S_n(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = 0$$

On the evidence of those results we infer, moreover, that

$$S_n(a,x,y,z) \text{ is a multinomial of order } 2n-1$$

from which it would follow that the number of terms that enter into the construction of $S_n(a,x,y,z)$ cannot exceed

$$\#_{5,2n-1} = \text{number of terms in } (1+a+x+y+z)^{2n-1}$$

where a combinatoric theorem familiar from the statistical mechanics of bosonic systems (proof immediate by the "stars and bars" argument) supplies

$$\begin{aligned}\#_{m,n} &= \text{number of terms in } (x_1+x_2+\cdots+x_m)^n\\ &= \binom{n+m-1}{m-1} = \binom{n+m-1}{n}\end{aligned}$$

From $\#_{5,5} = 126$, $\#_{5,7} = 330$ we see that fewer than one third of the anticipated terms actually enter into the construction of $S_3(a,x,y,z)$ and $S_4(a,x,y,z)$.

The results just developed are fruit of "Mayer's method," which, as previously remarked, permits one to circumvent altogether the difficulties that attend raising high-dimensional matrices to high powers and provides insight into how the resulting expressions *acquire* their complexly opaque structure. More to the point, Mayer's method permits one to address the *asymptotics* of the problem, concerning which the naive method provides not a clue.

Asymptotics of C-walks. The argument that gave (24) now gives

$$\begin{aligned}S_n(a,x,y,z) &\approx n\cdot \mathcal{S}(a,x,y,z)\\ \mathcal{S}(a,x,y,z) &= \frac{\sigma G_1 + (1-\tau)G_2 + G_3}{2+\sigma-\tau} = \frac{\mathcal{P}(a,x,y,z)}{\mathcal{Q}(a,x,y,z)}\end{aligned} \tag{25}$$

where

$$\begin{aligned}\mathcal{P} &= \text{multinomial of order 5, sum of 31 terms}\\ \mathcal{Q} &= \text{multinomial of order 4, sum of 21 terms}\end{aligned}$$

when spelled out in detail[13] read

$$\begin{aligned}\mathcal{P} &= 6(-1+3a-3a^2+a^3)\big[1-(x+y+z)+(xy+yz+zx)-xyz\big]\\ &\qquad + 6a^3xyz\\ \mathcal{Q} &= (3-6a+3a^2)-(2-5a+3a^2)(x+y+z)\\ &\qquad +(1-3a+3a^2)(xy+yz+zx)\end{aligned}$$

[13] Here I have made manifest the permutational xyz-symmetry of these asymptotic multinomials.

Confidence in the accuracy of these results is inspired by the observation that they give

$$\begin{aligned} S_n(a,a,a,a) &\approx n\frac{-6+36a-90a^2+120a^3-90a^4+36a^5}{3-12a+21a^2-18a^3+9a^4} \\ &= 2n(2a-1) \\ &= 2S_n(a) \end{aligned}$$

So here again (compare the point remarked at the end of the preceding section), the asymptotic formula is in such cases (which refer to the expected progress of a walker who takes n double steps, and advances with the same probability at each step) exact. Accuracy is supported also by the observations that

$$\begin{aligned} \mathcal{S}(0,0,0,0) &= -2 \\ \mathcal{S}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) &= 0 \\ \mathcal{S}(1,1,1,1) &= +2 \end{aligned}$$

which conform to the low-order exact results reported on the preceding page.

Parrondo's phenomenon...and its asymptotic extinction. Equations of the form $S_n(a,x,y,z)=0$—and asymptotically $\mathcal{S}(a,x,y,z)=0$—inscribe null surfaces within the unit 4-cube, and are not susceptible to graphic display except section by section. Which is perhaps the reason that Parrondo elected to set $y=z$, as we also will do. Walkers in such cases are subject to exceptional next-step probabilities only when they stand on lattice sites numbered $0 \bmod 3$.

Following Parrondo's lead, we undertake now to compare the walks generated by $\mathbb{A}=\mathbb{B}_{a,a,a}$, $\mathbb{B}=\mathbb{B}_{x,y,y}$ and $\mathbb{C}=\mathbb{C}_{a,x,y,y}=\mathbb{AB}$. Setting $y=z$ simplifies—but masks the informative symmetry of—the results achieved in preceding paragraphs. We find, for example, that (compare pages 8 & 15)

$$\begin{aligned} S_4(x,y,y) &= -4+4x+4y-2xy+2y^2+2xy^2-2y^3 \\ S_5(x,y,y) &= -5+4x+2x^3+6y-2xy+4x^2y-8x^3y+2y^2-6x^2y^2+8x^3y^2 \\ &\quad -6y^3+12xy^3-4x^2y^3+2y^4-4xy^4 \\ S_6(x,y,y) &= -6+4x+4x^2-2x^3+8y-14x^2y+8x^3y-4y^2+8xy^2+12x^2y^2 \\ &\quad -8x^3y^2+8y^3-12xy^3-8y^4+6xy^4+2y^5 \\ S_7(x,y,y) &= -7+6x+2x^4+8y-4xy+8x^3y-12x^4y+4y^2+4xy^2+6x^2y^2 \\ &\quad -32x^3y^2+24x^4y^2-4y^3-8xy^3+32x^3y^3-16x^4y^3-8y^4 \\ &\quad +36xy^4-30x^2y^4+8y^5-24xy^5+12x^2y^5-2y^6+4xy^6 \end{aligned}$$

$$\mathcal{S}(x,y,y) = \frac{-3+3(x+2y)-3(2xy+y^2)+6xy^2}{3-(x+2y)+(2xy+y^2)}$$

Graphic display indicates once again that the null curves defined $S_n(x,y,y)=0$ fall into two distinct classes according as n (small) is odd or even, and that members of both classes approach the curve $\mathcal{S}(x,y,y)=0$ as n becomes large:

see PLATE 3. Similarly (compare page 17—here I have proceeded by the naive method with 33-dimensional matrices—and page 18)

$$\begin{aligned}
S_2(a,x,y,y) &= -4+4a+2x+2ax+2x^2-4ax^2+2y-2ay-2xy+4axy\\
S_3(a,x,y,y) &= -6+6a+2x+2ax+2a^2x+2x^2+2ax^2-10a^2x^2+2x^3-8ax^3\\
&\quad +8a^2x^3+4y-2ay-2a^2y-2xy+8a^2xy-2x^2y+4ax^2y\\
&\quad -4a^2x^2y-2ay^2+2a^2y^2+4axy^2-4a^2xy^2\\
S_4(a,x,y,y) &= -8\ +\text{plus sum of 48 terms, of which the last is } +4a^3xy^3\\
S_5(a,x,y,y) &= -10+\text{plus sum of 94 terms, of which the last is } -4a^4xy^4\\
S_6(a,x,y,y) &= -12+\text{plus sum of 152 terms, of which the last is } +4a^5xy^5\\
S_7(a,x,y,y) &= -14+\text{plus sum of 235 terms, of which the last is } -4a^6xy^6\\
S_8(a,x,y,y) &= -16+\text{plus sum of 344 terms, of which the last is } +4a^7xy^7
\end{aligned}$$

$$\mathcal{S}(a,x,y,y) = \frac{6(-1+3a-3a^2+a^3)[1-(x+2y)+(2xy+y^2)-xy^2]+6a^3xy^2}{(3-6a+3a^2)-(2-5a+3a^2)(x+2y)+(1-3a+3a^2)(2xy+y^2)}$$

We turn now to the geometric implications of those analytic results. Familiarly, two linear transcepts of a square partition the square into four sectors (three if the lines intersect at the boundary). Similarly, three planar transcepts of a cube partition the cube into eight sectors ("octants" in the most familiar case). The null surfaces defined

$$\begin{aligned}
S_4(a) &= 0\\
S_4(x,y,y) &= 0\\
S_4(a,x,y,y) &= 0
\end{aligned}$$

unit cube in $\{a,x,y\}$-parameter space, and are seen in PLATE 4 to partition the cube into eight sectors, of which the hidden location of the eighth is revealed in PLATE 5. At points *interior* to those sectors $\{S_4(a), S_4(x,y,y), S_4(a,x,y,y)\}$—which, remember, describe the the expected mean 4-step progress of $\{A,B,C\}$ walkers, respectively—acquire *non-zero* values, and as one ranges over the sectors those appear with all $2^3 = 8$ possible sign combinations. The representative sector points indicated in PLATE 6 have coordinates

Sector	a	x	y
● (green)	0.495	0.20	0.83
● (blue)	0.495	0.70	0.10
● (red)	0.495	0.87	0.10
● (black)	0.495	0.98	0.10

at which the respective 4-step S-functions assume these values:

Sector	$S_{A,4}$	$S_{B,4}$	$S_{C,4}$
● (green)	−0.040	+0.298	−0.326
● (blue)	−0.040	−0.908	−0.450
● (red)	−0.040	−0.259	+0.453
● (black)	−0.040	+0.162	+1.064

Moving the location of the a-section to the other side of $a = \frac{1}{2}$ exposes the other four sectors; holding the $\{x, y\}$ parameters at their former values, we have

Sector	a	x	y
$\color{green}{\bullet\bullet}$	0.505	0.20	0.83
$\color{blue}{\bullet\bullet}$	0.505	0.70	0.10
$\color{red}{\bullet\bullet}$	0.505	0.87	0.10
$\color{black}{\bullet\bullet}$	0.505	0.98	0.10

Sector	$S_{A,4}$	$S_{B,4}$	$S_{C,4}$
$\color{green}{\bullet\bullet}$	+0.040	+0.298	−0.269
$\color{blue}{\bullet\bullet}$	+0.040	−0.908	−0.385
$\color{red}{\bullet\bullet}$	+0.040	−0.259	+0.498
$\color{black}{\bullet\bullet}$	+0.040	+0.162	+1.093

"Parrondo's phenomenon" is evident in sector $\color{red}{\bullet}$, where $S_{C,4}$ is positive though $S_{A,4}$ and $S_{B,4}$ are both negative (lose $*$ lose = win), and in sector $\color{green}{\bullet\bullet}$, where $S_{C,4}$ is negative though $S_{A,4}$ and $S_{B,4}$ are both positive (win $*$ win = lose). In all other sectors the sign of $S_{C,4}$ either (i) conforms to the shared sign of $S_{A,4}$ and $S_{B,4}$ or (ii) can be attributed to the predominance of $S_{A,4}$ else $S_{B,4}$.

As we ascend to higher orders we find, however (PLATE 7), that the volumes of the two "Parrondo sectors" is progressively reduced, and that in the asymptotic limit $n \to \infty$ (PLATE 8) they disappear altogether, evaporate without residue. This because the asymptotic null surfaces

$$\begin{aligned} \mathcal{S}(a) &= 0 \quad : \quad \text{entails } a = \tfrac{1}{2} \\ \mathcal{S}(x, y, y) &= 0 \\ \mathcal{S}(a, x, y, y) &= 0 \end{aligned}$$

intersect not on three curves but on a single/shared curve (and so partition the cube into not eight but only six sectors), as follows analytically from

$$\mathcal{P}(\tfrac{1}{2}, x, y, y) = \tfrac{3}{4}(-1 + x + 2y - 2xy - y^2 + 2xy^2) = \tfrac{1}{4}\mathcal{P}(x, y, y)$$

which is a special instance of the more general statement

$$\mathcal{P}(\tfrac{1}{2}, x, y, z) = \tfrac{1}{4}\mathcal{P}(x, y, z)$$

The asymptotic extinction of Parrondo's phenomenon (so far as it relates to $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C} = \mathbb{AB}$-generated walks on the unbounded lattice $\mathbb{Z}$) stands in stark contrast to the result reported in the literature,[6,7] and in contrast also to the result obtained when one looks (*Parrondo's Ratchet I*) to such walks on the cyclic graph of order 3. We have hit upon what might be called the "Parrondo's paradox paradox"—a circumstance that would appear to deprive Parrondo's discovery of much of its interest (particularly for those concerned with some of its conjectured applications, as to the development of investment strategies). Accepting the correctness on the one hand of the argument developed in the preceding pages, and the correctness on the other hand of the arguments developed in the literature, we are forced to the conclusion that the arguments in question address and resolve *distinct problems*. We acquire an obligation—which I will not attempt here to fulfill—to *identify the distinction*.

Generalizations and limitations. "Mayer's method" can in principle be used to develop the functions $S_n(x_1, x_2, \ldots, x_p)$ that pertain to walks generated by $\mathbb{B}$-matrices with period $p > 3$. But one would in such cases be led (see again (13) and (17)) to $p \times p$ Markovian matrices $\mathbb{Q}$ with spectra $\{1, \lambda_2, \ldots \lambda_p\}$ which generally defy analytic description (though it would remain an option to proceed case-by-case numerically). One is therefore prevented from writing closed-form analytic descriptions of the left/right eigenvectors and associated projection matrices $\{\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_p\}$. One is left with a set of attractive general formulæ to which one cannot ascribe specific meaning (except numerically, which is to say: pointwise, and pointwise information is not sufficient to support the "null surface" concept). And even if it were possible to write $S_n(x_1, x_2, \ldots, x_p) = 0$, graphic display requires sectioning that becomes ever more information-lossy as p increases beyond $p = 3$.

It is, of course, possible to contemplate walks (whether on the infinite lattice or on finite graphs) that are generated by more complex composite structures $\mathbb{C} = \mathbb{A}_1\mathbb{A}_2 \cdots \mathbb{A}_m$.

We have considered composite walks of the form $\mathbb{ABABAB} \cdots \mathbb{AB}$. The literature reports—and low-order naive-method experiments seem to confirm—that Perrondo's phenomenon persists even when the $\mathbb{A}$ and $\mathbb{B}$-generators are randomly rearranged. Here I point out that Mayer's method is so specific in its details—so "rigid"—as to be ill-adapted to exploration of the randomization question.

Acknowledgements. I owe everything to the inventive genius of Ray Mayer; my effort here has been simply to develop the detailed implications of his original idea. I am indebted also to David Griffiths for his editorial assistance and encouragement, and to the inspiring friendship of Harvey Leff.[3]

PLATE 1 (pages 5 & 12): Shown above are the qualitatively similar low-order null surfaces that arise from B-walks, solutions of

$$S_n(x, y, z) = 0 \quad : \quad n = 1, 3, 5, 7$$

The interleaved null surfaces that arise in even order are qualitatively distinct, and so similar to one another that in service of clarity I show only the surface of order 8. Colored spheres •, •, • mark unit points on the the x, y and z axes, respectively. A gray sphere • locates the origin; $S_n(x, y, z)$ is negative on the origin side, positive on the opposite side of the null surface. Common to all such surfaces is the scribed line (shown in red) at $x = \frac{1}{2}$.

PLATE 2 (page 14): The asymptotic surface that locates the solutions of

$$\mathcal{S}(x, y, z) = 0$$

which is well approximated already at order 8. The $\{x, y, z\}$-symmetry that is achieved asymptotically results now in three scribed lines (red) that partition the surface into two sets of three congruent sectors.

PLATE 3 (page 20): Shown above are the qualitatively similar low-order null curves that arise from B-walks with $z = y$, solutions of

$$S_n(x, y, y) = 0 \quad : \quad n = 1, 3, 5, 7$$

The null curves that arise in even order ($n = 2, 4, 6, 8$) comprise a qualitatively distinct population, shown below. The red curve arises from the asymptotic condition

$$\mathcal{S}(x, y, y) = 0$$

As before, •, •, • locate the origin and unit points on the x and y-axes.

PLATE 4 (page 20): The null surfaces that arise from 4-step A, B and C-walks are seen to partition the unit cube into 8 sectors.

PLATE 5 (page 20): The preceding figure, subjected now to the constraint $a \leqslant \frac{1}{2}$, in which four sectors are now more clearly evident. The orange sphere • marks the point $a = \frac{1}{2}$ on the a-axis, and the yellow sphere locates the center of the unit cube.

PLATE 6 (page 20): The preceding figure, sectioned at $a = 0.495$, with the solutions of

$$S_4(x, y, y) = 0 \quad \text{shown in blue}$$

$$S_4(0.495, x, y, y) = 0 \quad \text{shown in red}$$

Representative points used in the text are indicated by colored bullets. The red bullet • identifies the "Parrondo sector."

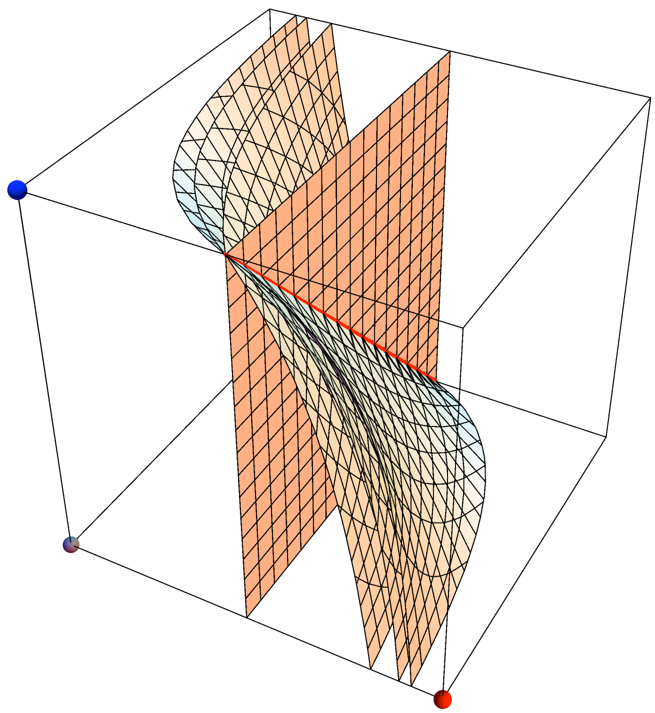
PLATE 7 (page 21): 8^{th}-order version of PLATE 5. Note the reduced volume of the "Parrondo sector."

PLATE 8 (page 21): Shown above, the null surfaces that arise from A, B and C-walks in the asymptotic limit—solutions of

$$\begin{aligned} \mathcal{S}(a) &= 0 \\ \mathcal{S}(x,y,y) &= 0 \\ \mathcal{S}(a,x,y,y) &= 0 \end{aligned}$$

The surfaces intersect on a single curve, and partition the unit cube (not into eight but) into only six sectors. Those facts are more vividly evident in the lower figure, where a has been constrained to the interval $[0, \frac{1}{2}]$. The curves shown in PLATE 6 have become coincident, squeezing the sectors marked • and • out of existence, with the consequence that *Parrondo's phenomenon has been extinguished*. Is it remarkable that this figure so closely resembles the figure displayed as PLATE 2 in *Parrondo's Ratchet I*?

PLATE 1



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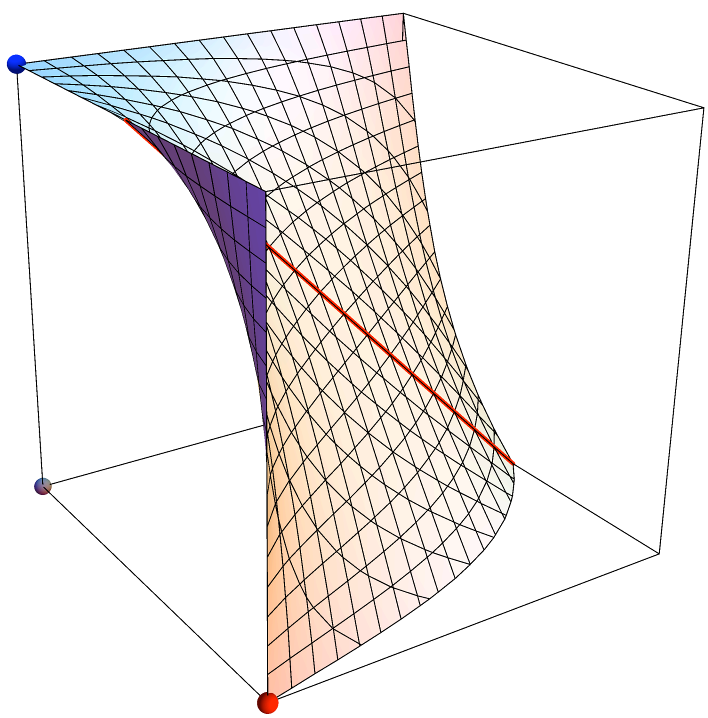


PLATE 2

Out[624]=

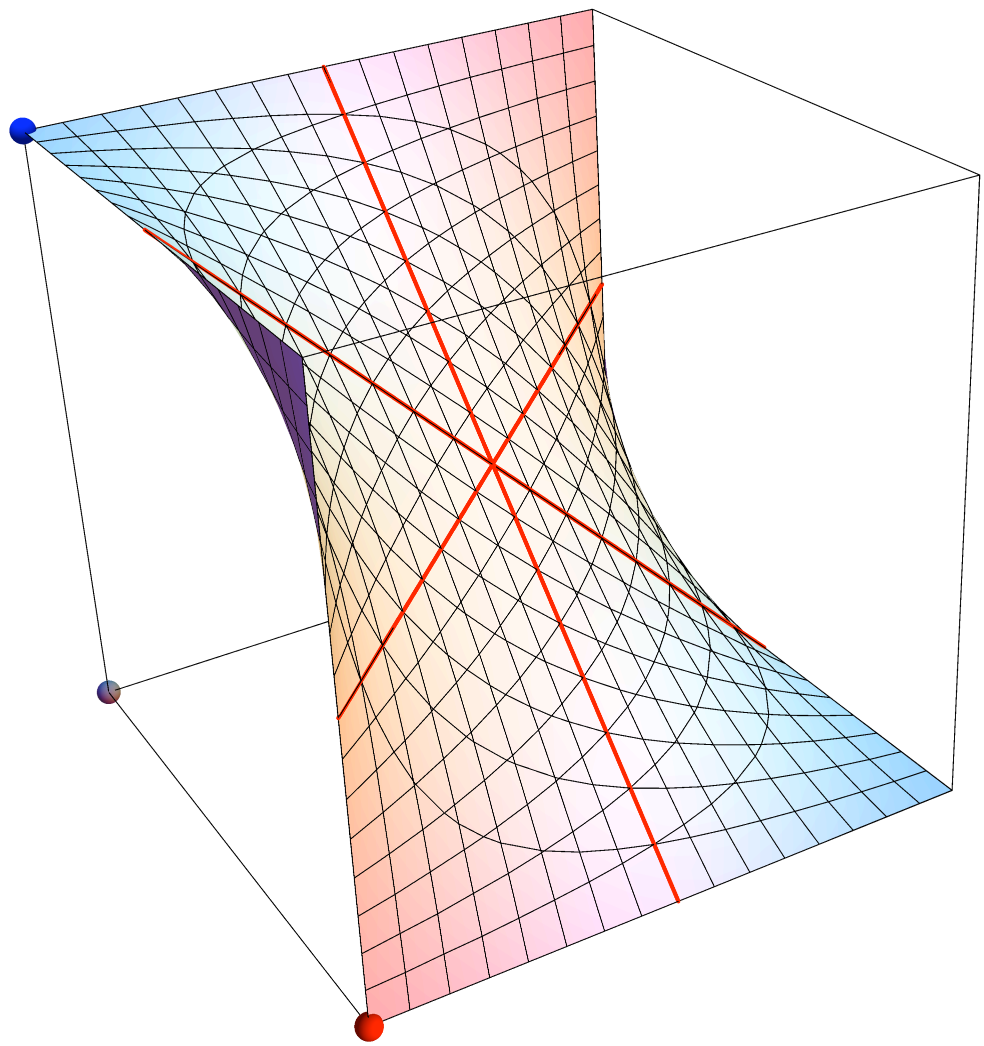
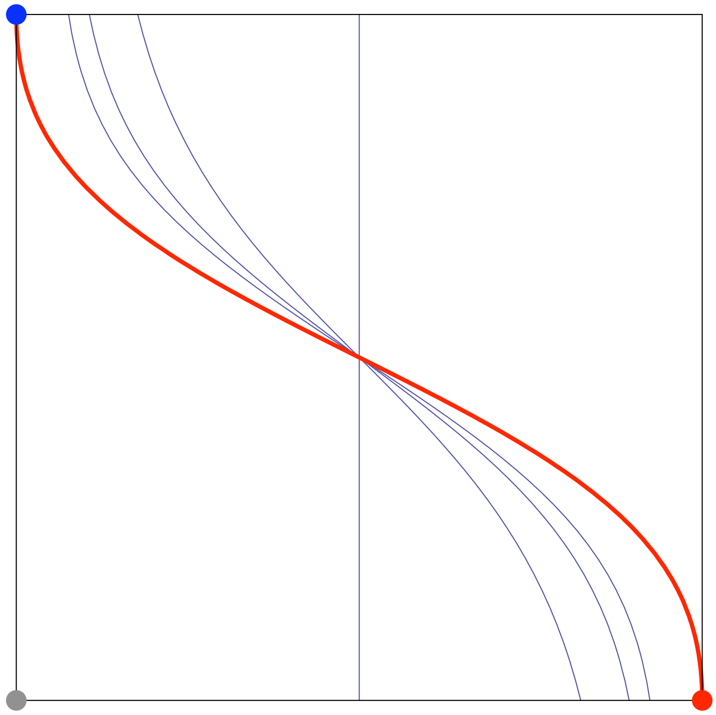


PLATE 3



Out[706]=

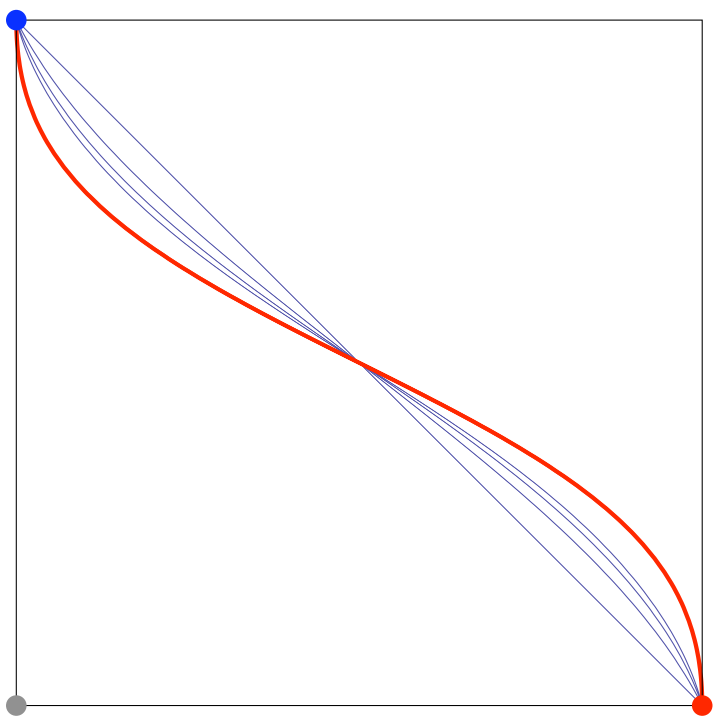


PLATE 4

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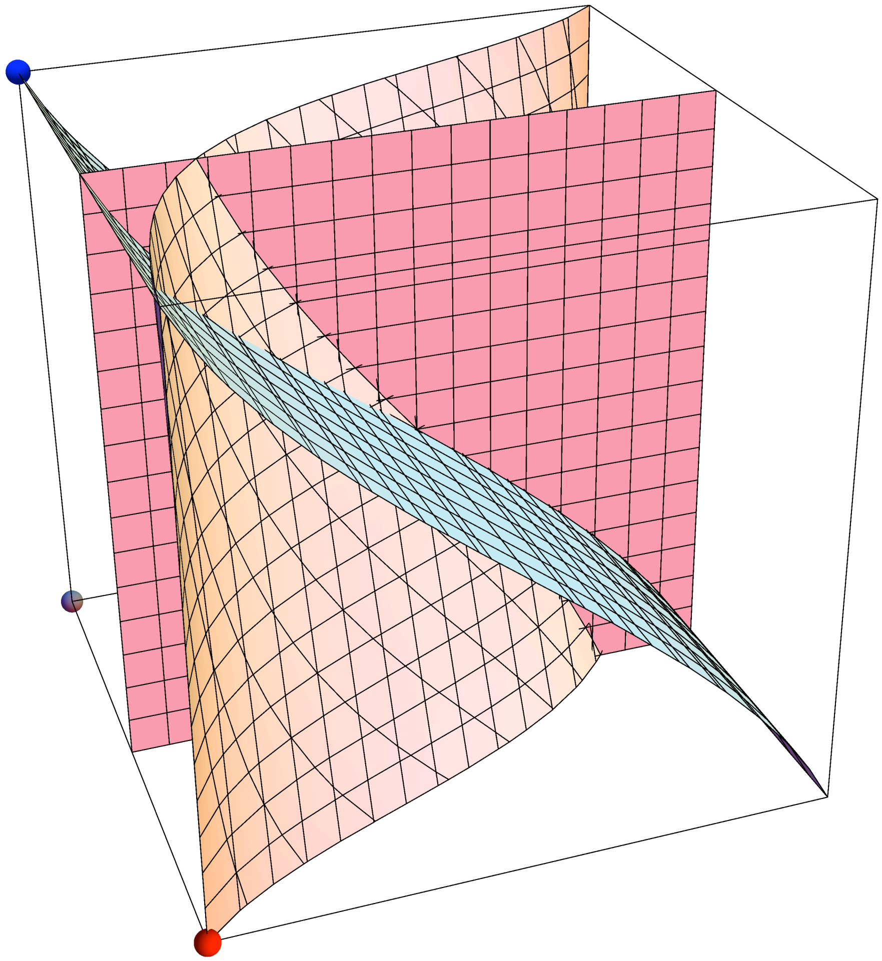
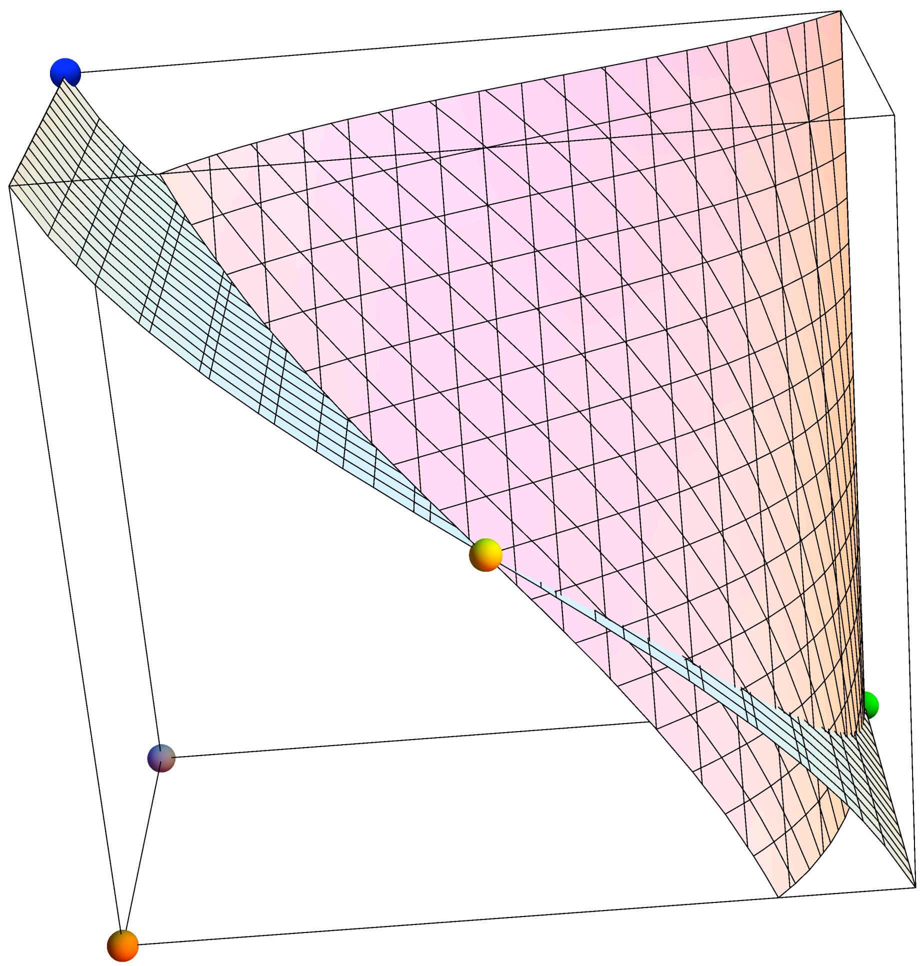


PLATE 5

Out[729]=



Out[791]=

PLATE 6

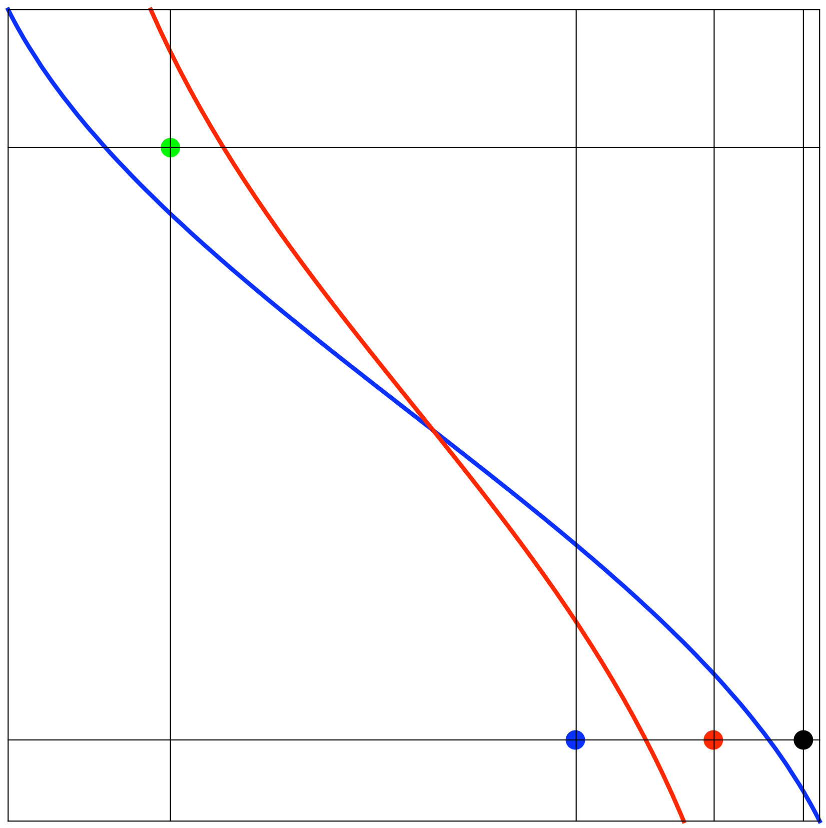


PLATE 7

Out[849]=

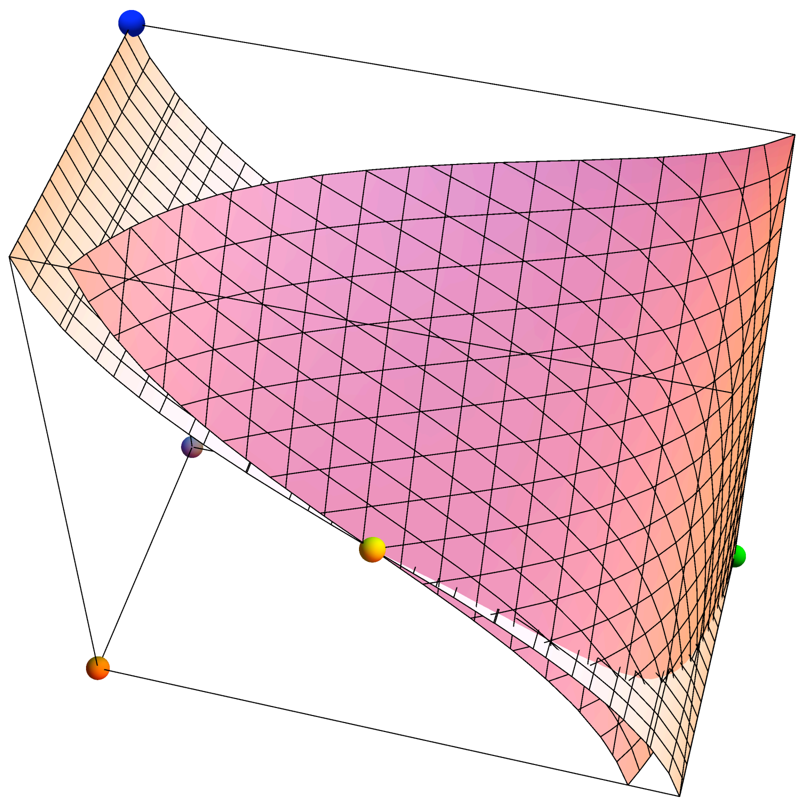
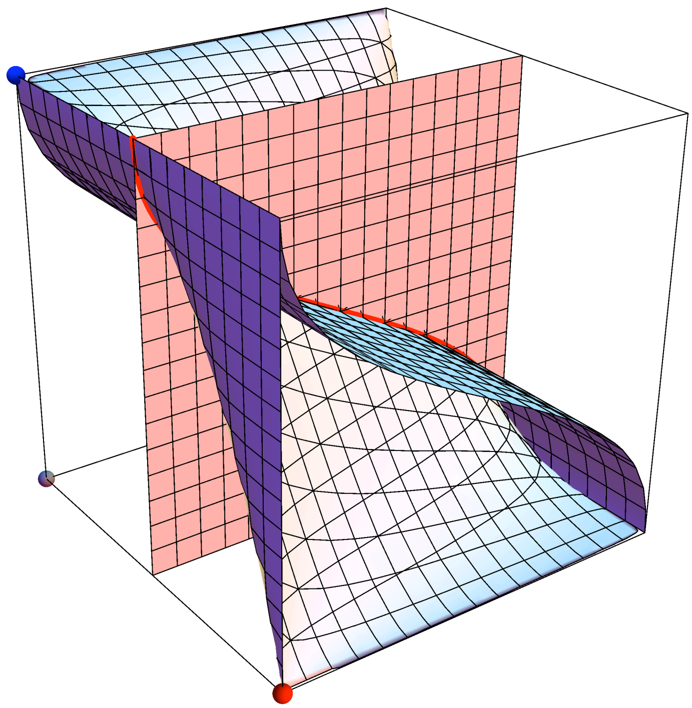


PLATE 8



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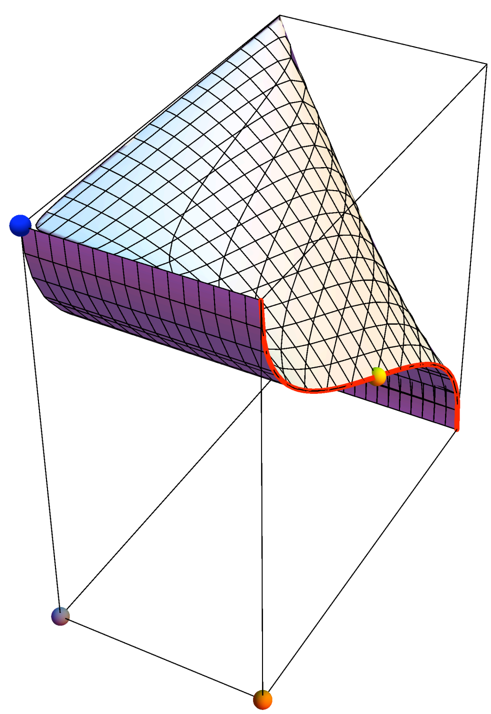

Remarks concerning the "Multinomial Similarity Problem." We have several times encountered lists[14] of multinomials of regularly ascending order and rapidly ascending length that when set equal to zero and plotted give rise to null curves/surfaces that are seen to be evermore similar and seemingly to approach limiting forms asymptotically. Our problem:[15] How to recognize/anticipate/account for that fact by examination of the multinomials themselves, without reference to the graphic evidence? The solution of that problem continues to elude me. I record here some elementary observations in the hope that they may prove relevant to its ultimate solution.

But Mayer's method led on pages 9-12 and 17 to constructions that permit one to display S_n as a sum of terms that are generated by an iterative process. I review how that comes about, taking B-walks as my illustrative context. We had

$$S_n = a_n G_1 + b_n G_2 + c_n G_3$$

where

$$\boldsymbol{h}_n \equiv \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \sum_{k=0}^{n-1} \mathbb{Q}^k \boldsymbol{h}_1 = \boldsymbol{h}_{n-1} + \mathbb{Q}^{n-1}\boldsymbol{h}_1 \quad : \quad n = 1, 2, 3, \ldots$$

with

$$\boldsymbol{h}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \mathbf{0}$$

Moreover,

$$\begin{aligned} \mathbb{Q}^{n-1}\boldsymbol{h}_1 &= \mathbb{P}\boldsymbol{h}_1 + \big(\lambda_2^{n-1}\mathbb{P}_2 + \lambda_3^{n-1}\mathbb{P}_3\big)\boldsymbol{h}_1 \\ &\equiv \;\; \boldsymbol{r} \;\;\; + \;\;\; \boldsymbol{s}_{n-1} \end{aligned}$$

which gives

$$\begin{aligned} \boldsymbol{h}_1 &= \;\;\boldsymbol{r} + \boldsymbol{s}_0 \\ \boldsymbol{h}_2 &= 2\,\boldsymbol{r} + \boldsymbol{s}_0 + \boldsymbol{s}_1 \\ \boldsymbol{h}_3 &= 3\,\boldsymbol{r} + \boldsymbol{s}_0 + \boldsymbol{s}_1 + \boldsymbol{s}_2 \\ &\vdots \\ \boldsymbol{h}_n &= n\boldsymbol{r} + \boldsymbol{s}_0 + \boldsymbol{s}_1 + \cdots + \boldsymbol{s}_{n-1} \end{aligned} \tag{26}$$

For B-walks we find

$$\boldsymbol{r} = \frac{1}{2+\sigma}\begin{pmatrix} \sigma \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{s}_0 = \frac{1}{2+\sigma}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \boldsymbol{s}_1 = \frac{1}{2+\sigma}\begin{pmatrix} -\sigma \\ 1+\sigma \\ -1 \end{pmatrix}, \quad \boldsymbol{s}_2 = \frac{1}{2+\sigma}\begin{pmatrix} -\sigma \\ -1 \\ 1+\sigma \end{pmatrix}$$

$$\boldsymbol{s}_3 = \frac{1}{2+\sigma}\begin{pmatrix} \sigma(1+\sigma) \\ 1-\sigma-\sigma^2 \\ -1 \end{pmatrix}, \quad \boldsymbol{s}_4 = \frac{1}{2+\sigma}\begin{pmatrix} -\sigma \\ -1+2\sigma+\sigma^2 \\ 1-\sigma-\sigma^2 \end{pmatrix}, \quad \textit{etc.}$$

---

[14] See pages 4-5, 12, 13, 17, 19 and 20.

[15] See again the second of the bulleted questions on page 5.

which when introduced into (26) are found to give back precisely the $\boldsymbol{h}$-vectors described on page 12. From which—magically—all the $(2+\sigma)$-denominators have canceled.

The elements of $\boldsymbol{r}$ led on page 13 to the construction of the asymptotic function

$$\mathcal{S} = \frac{\sigma G_1 + G_2 + G_3}{2+\sigma}$$

Let $\mathcal{S}_k$ denote the function that results similarly from the coordinates of $\boldsymbol{s}_k$. We then have

$$S_n = n\,\mathcal{S} + \mathcal{S}_0 + \mathcal{S}_1 + \cdots + \mathcal{S}_{n-1} \tag{27}$$

From $|\lambda_2| < 1$ and $|\lambda_3| < 1$ we infer that successive terms in (27) become progressively less significant. On those same grounds we have

$$\boldsymbol{t} = \sum_{k=0}^{\infty} \boldsymbol{s}_k = \Big(\frac{1}{1-\lambda_2}\mathbb{P}_2 + \frac{1}{1-\lambda_3}\mathbb{P}_3\Big)\boldsymbol{h}_1 = \frac{1}{(2+\sigma)^2}\begin{pmatrix} 4-\sigma \\ -1+\sigma \\ -3 \end{pmatrix}$$

giving

$$\sum_{k=0}^{\infty} \mathcal{S}_k = \mathcal{T}$$

$$\mathcal{T} = \frac{(4-\sigma)G_1 + (\sigma-1)G_2 - 3G_3}{(2+\sigma)^2}$$

While the functions $S_n(x,y,z)$ are multinomials of ascending order, the functions $\mathcal{S}_k(x,y,z)$ are *ratios* of multinomials (numerators of ascending order, denominators of fixed low order), as also are the functions $\mathcal{S}(x,y,z)$ and $\mathcal{T}(x,y,z)$ (numerators and denominators both of fixed low order). By Taylor expansion those ratios can be displayed as multinomials of infinite order; we at (12) had

$$\sigma = 1 - (x+y+z) + (xy+yz+zx)$$

and by graphic experimentation[16] establish that $0 < \sigma < 1$ as $\{x,y,z\}$ ranges on the interior of the unit cube. So we have

$$\frac{1}{2+\sigma} = \tfrac{1}{2} - \tfrac{1}{4}\sigma + \tfrac{1}{8}\sigma^2 - \tfrac{1}{16}\sigma^3 + \tfrac{1}{32}\sigma^4 + \cdots$$

giving (for example)

$$\mathcal{S} = \big(\sigma G_1 + G_2 + G_3\big)\cdot\big(\tfrac{1}{2} - \tfrac{1}{4}\sigma + \tfrac{1}{8}\sigma^2 - \tfrac{1}{16}\sigma^3 + \tfrac{1}{32}\sigma^4 + \cdots\big)$$

Here the leading factor dictates the form of the asymptotic null surface, while the elements of the second factor serve to refine the off-surface values of $\mathcal{S}$.

The results developed above render explicit the origin of the covert "similarity structure" implicit in the S-multinomials that were spelled out on pages 4 and 5 (similar remarks pertain to the multinomials that on page 17 were derived from C-walks), but they provide no indication of how that structure might be deduced from direct inspection of the multinomials themselves.

[16] Use `ContourPlot3D[`$\sigma == \sigma_0, \{x,0,1\}, \{y,0,1\}, \{z,0,1\}$`]` for assorted values of $\sigma_0 \in [0,1]$.